Chapter 15

THE BÉZIER CURVE

The Bézier curve is an important part of almost every computer-graphics illustration program and computer-aided design system in use today. It is used in many ways, from designing the curves and surfaces of automobiles to defining the shape of letters in type fonts. And because it is numerically the most stable of all the polynomial-based curves used in these applications, the Bézier curve is the ideal standard for representing the more complex piecewise polynomial curves.

In the early 1960s, Peter Bézier (pronounced *bay-zee-aye*) began looking for a better way to define curves and surfaces, one that would be useful to a design engineer. He was familiar with the work of Ferguson and Coons and their parametric cubic curves and bicubic surfaces. However, these did not offer an intuitive way to alter and control shape. The results of Bézier's research led to the curves and surfaces that bear his name and became part of the UNISURF system. The French automobile manufacturer, Renault, used UNISURF to design the sculptured surfaces of many of its products.

This chapter begins by describing a surprisingly simple geometric construction of a Bézier curve, followed by a derivation of its algebraic definition, basis functions, control points, degree elevation, and truncation. It concludes by showing how to join two curves end-to-end to form a single composite curve.

15.1 A Geometric Construction

We can draw a Bézier curve using a simple recursive geometric construction. Let's begin by constructing a second-degree curve (Figure 15.1). We select three points A, B, C, so that line AB is tangent to the curve at A, and line BC is tangent at C. The curve begins at A and ends at C. For any ratio u_i, where $0 \leq u_i \leq 1$, we construct points D and E so that

$$\frac{AD}{AB} = \frac{BE}{BC} = u_i \tag{15.1}$$

On DE we construct F so that $DF/DE = u_i$. Point F is on the curve.

Repeating this process for other values of u_i, we produce a series of points on a Bézier curve. Note that we must be consistent in the order in which we sub-divide AB and BC. For example, $AD/AB \neq EC/BC$.

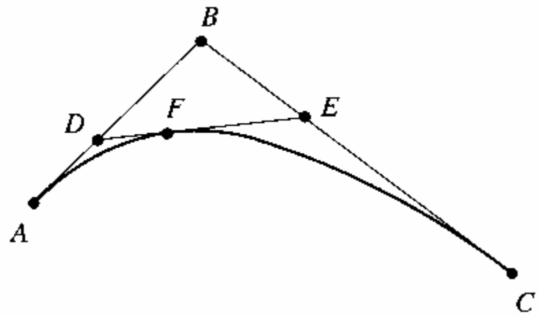


Figure 15.1 *Geometric construction of a second-degree Bézier curve.*

To define this curve in a coordinate system, let point $A = x_A, y_A$, $B = x_B, y_B$, and $C = x_C, y_C$. Then coordinates of points D and E for some value of u_i are

$$\begin{aligned} x_D &= x_A + u_i(x_B - x_A) \\ y_D &= y_A + u_i(y_B - y_A) \end{aligned} \tag{15.2}$$

and

$$\begin{aligned} x_E &= x_B + u_i(x_C - x_B) \\ y_E &= y_B + u_i(y_C - y_B) \end{aligned} \tag{15.3}$$

The coordinates of point F for some value of u_i are

$$\begin{aligned} x_F &= x_D + u_i(x_E - x_D) \\ y_F &= y_D + u_i(y_E - y_D) \end{aligned} \tag{15.4}$$

To obtain x_F and y_F in terms of the coordinates of points A, B, and C, for any value of u_i in the unit interval, we substitute appropriately from Equations 15.2 and 15.3 into Equations 15.4. After rearranging terms to simplify, we find

$$\begin{aligned} x_F &= (1-u_i)^2 x_A + 2u_i(1-u_i)x_B + u_i^2 x_C \\ y_F &= (1-u_i)^2 y_A + 2u_i(1-u_i)y_B + u_i^2 y_C \end{aligned} \tag{15.5}$$

We generalize this set of equations for any point on the curve using the following substitutions:

$$\begin{aligned} x(u) &= x_F \\ y(u) &= y_F \end{aligned} \tag{15.6}$$

and we let

$$\begin{aligned} x_0 &= x_A \quad x_1 = x_B \quad x_2 = x_C \\ y_0 &= y_A \quad y_1 = y_B \quad y_2 = y_C \end{aligned} \tag{15.7}$$

Now we can rewrite Equation 15.5 as

$$\begin{aligned} x(u) &= (1-u)^2 x_0 + 2u(1-u)x_1 + u^2 x_2 \\ y(u) &= (1-u)^2 y_0 + 2u(1-u)y_1 + u^2 y_2 \end{aligned} \tag{15.8}$$

This is the set of second-degree equations for the coordinates of points on a Bézier curve, based on our construction.

We express this construction process and Equations 15.8 in terms of vectors with the following substitutions: Let the vector $\mathbf{p}_0$ represent point A, $\mathbf{p}_1$ point B, and $\mathbf{p}_2$ point C. From vector geometry we have $D \equiv \mathbf{p}_0 + u(\mathbf{p}_1 - \mathbf{p}_0)$, and $E \equiv \mathbf{p}_1 + u(\mathbf{p}_2 - \mathbf{p}_1)$. If we let $F \equiv \mathbf{p}(u)$, we see that

$$\mathbf{p}(u) = \mathbf{p}_0 + u(\mathbf{p}_1 - \mathbf{p}_0) + u[\mathbf{p}_1 + u(\mathbf{p}_2 - \mathbf{p}_1) - \mathbf{p}_0 + u(\mathbf{p}_1 - \mathbf{p}_0)] \tag{15.9}$$

We rearrange terms to obtain a more compact vector equation of a second-degree Bézier curve:

$$\mathbf{p}(u) = (1-u)^2\mathbf{p}_0 + 2u(1-u)\mathbf{p}_1 + u^2\mathbf{p}_2 \tag{15.10}$$

where the points $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ are called *control points*, and $0 \leq u \leq 1$. The ratio u is the *parametric variable*. Later, we will see that this equation is an example of a *Bernstein polynomial*. Note that the curve will always lie in the plane containing the three control points, but the points do not necessarily lie in the xy plane.

Similar constructions apply to Bézier curves of any degree. In fact the degree of a Bézier curve is equal to $n-1$, where n is the number of control points.

Figure 15.2 shows the construction of a point on a cubic Bézier curve, which requires four control points A, B, C, and D to define it. The curve begins at point A tangent to line AB, and ends at D and tangent to CD. We construct points E, F, and G so that

$$\frac{AE}{AB} = \frac{BF}{BC} = \frac{CG}{CD} = u_i \tag{15.11}$$

On EF and FG we locate H and I, respectively, so that

$$\frac{EH}{EF} = \frac{FI}{FG} = u_i \tag{15.12}$$

Finally, on HI we locate J so that

$$\frac{HJ}{HI} = u_i \tag{15.13}$$

We can make no more subdivisions, which means that point J is on the curve. If we continue this process for a sequences of points, then their locus defines the curve.

If points A, B, C, and D are represented by the vectors $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, respectively, then expressing the construction of the intermediate points E, F, G, H, and

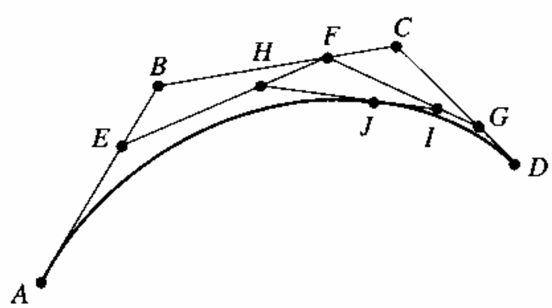


Figure 15.2 *Geometric construction of a cubic Bézier curve.*

I in terms of these vectors to produce point J, or $\mathbf{p}(u)$, yields

$$\begin{aligned}\mathbf{p}(u) = {} & \mathbf{p}_0 + u(\mathbf{p}_1 - \mathbf{p}_0) + u[\mathbf{p}_1 + u(\mathbf{p}_2 - \mathbf{p}_1) - \mathbf{p}_0 - u(\mathbf{p}_1 - \mathbf{p}_0)] \\ & + u\{\mathbf{p}_1 + u(\mathbf{p}_2 - \mathbf{p}_1) + u[\mathbf{p}_2 + u(\mathbf{p}_3 - \mathbf{p}_2) - \mathbf{p}_1 - u(\mathbf{p}_2 - \mathbf{p}_0)] - \mathbf{p}_0 \\ & - u(\mathbf{p}_1 - \mathbf{p}_0) - u[\mathbf{p}_1 + u(\mathbf{p}_2 - \mathbf{p}_1) - \mathbf{p}_0 - u(\mathbf{p}_1 - \mathbf{p}_0)]\}\end{aligned} \tag{15.14}$$

This awkward expression simplifies nicely to

$$\mathbf{p}(u) = (1-u)^3\mathbf{p}_0 + 3u(1-u)^2\mathbf{p}_1 + 3u^2(1-u)\mathbf{p}_2 + u^3\mathbf{p}_3 \tag{15.15}$$

Of course, this construction of a cubic curve with its four control points is done in the plane of the paper. However, the cubic polynomial allows a curve that is nonplanar; that is, it can represent a curve that twists in space.

The geometric construction of a Bézier curve shows how the control points influence its shape. The curve begins on the first point and ends on the last point. It is tangent to the lines connecting the first two points and the last two points. The curve is always contained within the *convex hull* of the control points.

No one spends time constructing and plotting the points of a Bézier curve by hand, of course. A computer does a much faster and more accurate job. However, it is worth doing several curves this way for insight into the characteristics of Bézier curves.

15.2 An Algebraic Definition

Bézier began with the idea that any point $\mathbf{p}(u)$ on a curve segment should be given by an equation such as the following:

$$\mathbf{p}(u) = \sum_{i=0}^{n} \mathbf{p}_i f_i(u) \tag{15.16}$$

where $0 \le u \le 1$, and the vectors $\mathbf{p}_i$ are the control points (Figure 15.3).

Equation 15.16 is a compact way to express the sum of several similar terms, because what it says is this:

$$\mathbf{p}(u) = \mathbf{p}_0 f_0(u) + \mathbf{p}_1 f_1(u) + \cdots + \mathbf{p}_n f_n(u) \tag{15.17}$$

of which Equation 15.10 and 15.15 are specific examples, for $n = 2$ and $n = 3$, respectively.

The $n + 1$ functions, that is the $f_i(u)$, must produce a curve that has certain well-defined characteristics. Here are some of the most important ones:

1. The curve must start on the first control point, $\mathbf{p}_0$, and end on the last, $\mathbf{p}_n$. Mathematically, we say that the functions must interpolate these two points.
2. The curve must be tangent to the line given by $\mathbf{p}_1 - \mathbf{p}_0$ at $\mathbf{p}_0$ and to $\mathbf{p}_n - \mathbf{p}_{n-1}$ at $\mathbf{p}_n$.
3. The functions $f_i(u)$ must be symmetric with respect to u and $(1 - u)$. This lets us reverse the sequence of control points without changing the shape of the curve.

Other characteristics can be found in more advanced works on this subject.

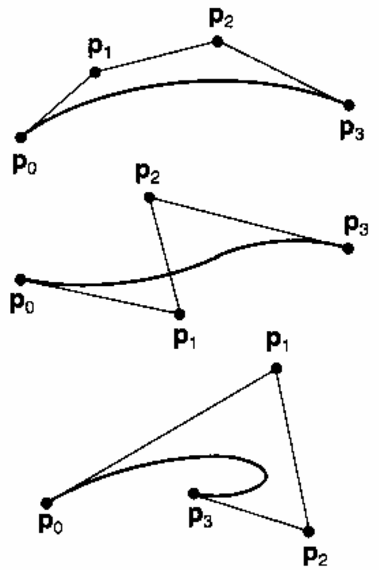


Figure 15.3 *Bézier curves and their control points.*

A family of functions called *Bernstein polynomials* satisfies these requirements. They are the *basis functions* of the Bézier curve. (Other curves, such as the *NURBS curves*, use different, but related, basis functions.) We rewrite Equation 15.16 using them, so that

$$\mathbf{p}(u) = \sum_{i=0}^{n} \mathbf{p}_i B_{i,n}(u) \tag{15.18}$$

where the basis functions are

$$B_{i,n}(u) = \binom{n}{i} u^i (1-u)^{n-i} \tag{15.19}$$

The term $\binom{n}{i}$ is the binomial coefficient function from probability theory and statistics, defined as

$$\binom{n}{i} = \frac{n!}{i!(n-1)!} \tag{15.20}$$

The symbol ! is the factorial operator. For example, $3! = 3 \times 2 \times 1, 5! = 5 \times 4 \times 3 \times 2 \times 1$, and so forth. We use the following conventions when evaluating Equation 15.20: If i and u equal zero, then $u^i = 1$ and $0! = 1$. We see that for $n+1$ control points, the basis functions produce an nth-degree polynomial.

Expanding Equation 15.18 for a second-degree Bézier curve (where $n = 2$ and there are three control points) produces

$$\mathbf{p}(u) = \mathbf{p}_0 B_{0,2}(u) + \mathbf{p}_1 B_{1,2}(u) + \mathbf{p}_2 B_{2,2}(u) \tag{15.21}$$

From Equation 15.20 we find

$$B_{0,2}(u) = (1-u)^2 \tag{15.22}$$

$$B_{1,2}(u) = 2u(1-u) \tag{15.23}$$

$$B_{2,2}(u) = u^2 \tag{15.24}$$

These are the basis functions for a second-degree Bézier curve. Substituting them into Equation 15.21 and rearranging terms, we find

$$\mathbf{p}(u) = (1-u)^2\mathbf{p}_0 + 2u(1-u)\mathbf{p}_1 + u^2\mathbf{p}_2 \tag{15.25}$$

This is the same expression we found from the geometric construction, Equation 15.10. The variable u is now called the *parametric variable.*

Now, let's expand Equation 15.18 for a cubic Bézier curve, where $n = 3$:

$$\mathbf{p}(u) = \mathbf{p}_0 B_{0,3}(u) + \mathbf{p}_1 B_{1,3}(u) + \mathbf{p}_2 B_{2,3}(u) + \mathbf{p}_3 B_{3,3}(u) \tag{15.26}$$

and from Equation 15.20 we find

$$B_{0,3}(u) = (1-u)^3 \tag{15.27}$$

$$B_{1,3}(u) = 3u(1-u)^2 \tag{15.28}$$

$$B_{2,3}(u) = 3u^2(1-u) \tag{15.29}$$

$$B_{3,3}(u) = u^3 \tag{15.30}$$

Substituting these into Equation 15.26 and rearranging terms produces

$$\mathbf{p}(u) = (1-u)^3\mathbf{p}_0 + 3u(1-u)^2\mathbf{p}_1 + 3u^2(1-u)\mathbf{p}_2 + u^3\mathbf{p}_3 \tag{15.31}$$

Bézier curve equations are well suited for expression in matrix form. We can expand the cubic parametric functions and rewrite Equation 15.31 as

$$\mathbf{p}(u) = \begin{bmatrix} (1-3u+3u^2-u^3) \\ (3u-6u^2+3u^3) \\ (3u^2-3u^3) \\ u^3 \end{bmatrix}^T \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \tag{15.32}$$

or as

$$\mathbf{p}(u) = [u^3 \quad u^2 \quad u \quad 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \tag{15.33}$$

If we let

$$\mathbf{U} = [u^3 \quad u^2 \quad u \quad 1] \tag{15.34}$$

$$\mathbf{P} = [\mathbf{p}_0 \quad \mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3]^T \tag{15.35}$$

and

$$\mathbf{M} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{15.36}$$

then we can write Equation 15.33 even more compactly as

$$\mathbf{p}(u) = \mathbf{UMP} \tag{15.37}$$

Note that the composition of the matrices **U**, **M**, and **P** varies according to the number of control points (that is, the degree of the Bernstein polynomial basis functions).

15.3 Control Points

In Equation 15.18, we see that the control points are coefficients of the Bernstein polynomial basis functions. Connecting the control points in sequence with straight lines yields the Bézier *control polygon*, and the curve lies entirely within its convex hull. The control polygon establishes the initial shape of a curve, and also crudely approximates this shape. It also gives us a way to change a curve's shape. Again, notice that $n+1$ control points produce an nth-degree curve.

Figure 15.4 shows how different sequences of three identical sets of points affect the shape of the curve. The order in Figure 15.4a is reversed in Figure 15.4b and curve shape was not affected. Reversing the control point sequence reverses the direction of parameterization. In Figure 15.4c, the sequence is more dramatically altered, creating an altogether different curve. Note, though, that the curve is always tangent to the first and last edges of the control polygon.

Studying the basis functions helps us to understand curve behavior. Figure 15.5 shows basis function plots over the unit interval for cubic Bézier curves. The contribution of the first control point, $\mathbf{p}_0$, is propagated throughout the curve by $B_{0,3}$, and it is most influential at $u = 0$. The other control points do not contribute to $\mathbf{p}(u)$ at $u = 0$, because

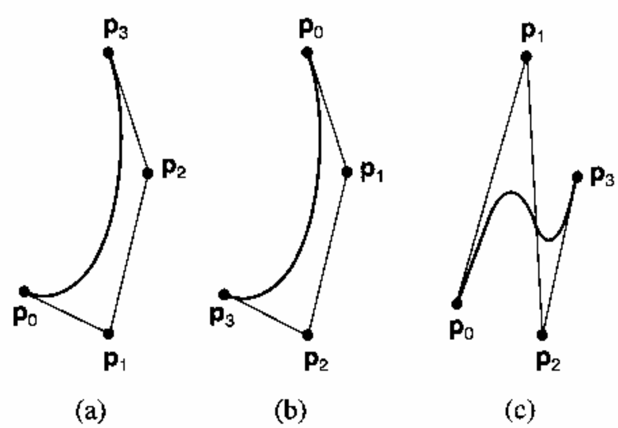


Figure 15.4 *Three different sequences of four control points.*

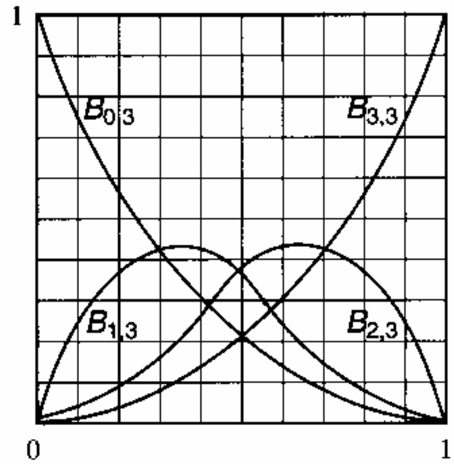


Figure 15.5 *Bézier curve basis functions.*

$B_{0,3}(u) = 1$ and $B_{1,3}(0) = B_{2,3}(0) = B_{3,3}(0) = 0$. Control point $\mathbf{p}_1$ is most influential at $u = 1/3$, and $\mathbf{p}_2$ at $u = 2/3$. At $u = 1$, only $\mathbf{p}_3$ affects $\mathbf{p}(u)$. Note the symmetry of $B_{0,1}(u)$ and $B_{3,3}(u)$, as well as that of $B_{1,3}(u)$ and $B_{2,3}(u)$.

We see that the effect of any control point is weighted by its associated basis function. This means that if we change the position of a control point $\mathbf{p}_i$, the greatest effect on the curve shape is at or near the parameter value $u = i/n$.

Figure 15.6 shows two examples of how we can modify the shape of a Bézier curve. In Figure 15.6a, moving $\mathbf{p}_2$ to $\mathbf{p}'_2$ pulls the curve toward that point. In Figure 15.6b, we add one or two extra control points at $\mathbf{p}_1$. The multiply-coincident points pull the curve closer and closer to that vertex. In this case, each additional control point raises the degree of the basis function polynomials.

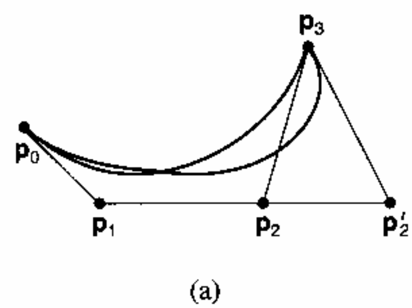
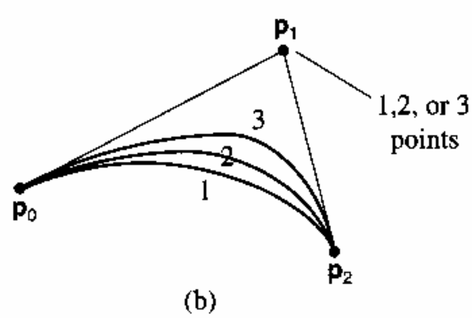


Figure 15.6 *Modifying the shape of a Bézier curve.*



We asserted previously that we could traverse a set of control points in either direction without affecting shape. They could be ordered $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ or $\mathbf{p}_n, \mathbf{p}_{n-1}, \ldots, \mathbf{p}_0$. The curve is the same, and only the direction of parameterization is reversed. We express this equivalence as

$$\sum_{i=0}^{n} \mathbf{p}_i B_{i,n}(u) = \sum_{i=0}^{n} \mathbf{p}_{n-i} B_{i,n}(1-u) \tag{15.38}$$

which follows from the basis function identity

$$B_{i,n}(u) = B_{n-i,n}(1-u) \tag{15.39}$$

15.4 Degree Elevation

Each control point we add to the definition of a Bézier curve raises its degree by one. We might choose to do this if we are not satisfied with a curve's original shape or the possible shapes available to us by moving any of the original control points. Usually it is advisable to add another point in a way that does not initially change the shape of the curve (Figure 15.7). After we have added a point, we can move it or any of the other control points to change the shape of the curve. If a new set of control points ${}^1\mathbf{p}_i$ generates the same curve as the original set $\mathbf{p}_i$, then it follows that

$$\sum_{i=0}^{n+1} {}^1\mathbf{p}_i B_{i,n+1}(u) = \sum_{i=0}^{n} \mathbf{p}_i B_{i,n}(u) \tag{15.40}$$

or

$$\sum_{i=0}^{n+1} {}^1\mathbf{p}_i \binom{n+1}{i} u^i (1-u)^{n+1-i} = \sum_{i=0}^{n} \mathbf{p}_i \binom{n}{i} u^i (1-u)^{n-i} \tag{15.41}$$

Equation 15.41 is the result of substituting Equation 15.19 into Equation 15.18. If we multiply the right side of Equation 15.41 by $u + (1 - u)$, we obtain

$$\sum_{i=0}^{n+1} {}^1\mathbf{p}_i \binom{n+1}{i} u^i (1-u)^{n+1-i} = \sum_{i=0}^{n} \mathbf{p}_i \binom{n}{i} \left[u^i (1-u)^{n+1-i} + u^{i+1}(1-u)^{n-i}\right] \tag{15.42}$$

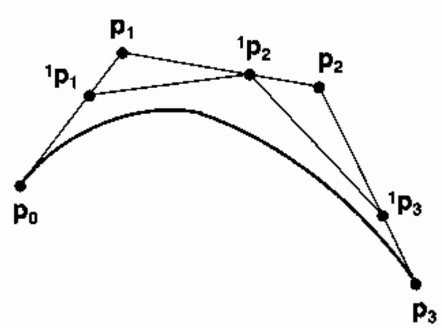


Figure 15.7 *Degree elevation: adding a control point that does not initially change the shape of the curve.*

Summing the right side of this equation produces $2(n+1)$ terms, and summing the left side produces $n+2$ terms. By rearranging and grouping terms on the right side so that we can compare and equate the coefficients of $u^i(1-u)^{n+1-i}$ on both sides, we can write

$$ {}^1\mathbf{p}_i \binom{n+1}{i} = \mathbf{p}_i \binom{n}{i} + \mathbf{p}_{i-1} \binom{n}{i-1} \tag{15.43} $$

Next, we expand the binomial coefficient terms and simplify, to obtain

$$ {}^1\mathbf{p}_i = \left(\frac{i}{n+1}\right)\mathbf{p}_{i-1} + \left(1 - \frac{i}{n+1}\right)\mathbf{p}_i \tag{15.44} $$

for $i = 0, 1, \ldots, n+1$.

Equation 15.44 says that we can compute a new set of control points ${}^1\mathbf{p}_i$ from the original points. Figure 15.7 shows what happens when we add a point to a cubic Bézier curve. Note that the new interior points fall on the sides of the original control polygon. We can repeat this process until we have added enough control points to satisfactorily control the curve's shape.

15.5 Truncation

If we want to retain only a part of a Bézier curve, for example, the curve segment between u_i and u_j, then we must truncate the segments from $u = 0$ to $u = u_i$ and from $u = u_j$ to $u = 1$ (Figure 15.8). We must find the set of control points that defines the segment that remains. To do this we must change the parametric variable so that it again varies over the unit interval, instead of over $u_i \le u \le u_j$, as in the original curve. We introduce a new parameter v, where

$$ v = au + b \tag{15.45} $$

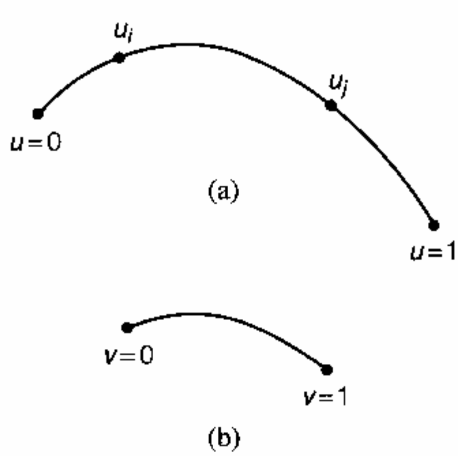


Figure 15.8 *Truncating a Bézier curve.*

This transformation applies to parametric polynomial equations of any degree. The linear relationship between u and v (Equation 15.45) preserves the degree of the polynomial.

If $v = 0$ at $u = -u_i$, and $v = 1$ at $u = u_j$, then

$$u = (u_j - u_i)v + u_i \tag{15.46}$$

or

$$u = \Delta u_i v + u_i \tag{15.47}$$

where

$$\Delta u_i = u_j - u_i \tag{15.48}$$

The general transformation equation for the parametric variable (not derived here) is

$$u^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} u_i^k (\Delta u_i v)^{n-k} \tag{15.49}$$

This equation looks more formidable than it is. From it we extract a transformation matrix **T** such that

$$[u^n \quad u^{n-1} \quad \cdots \quad u \quad 1] = [v^n \quad v^{n-1} \quad \cdots \quad v \quad 1]\mathbf{T} \tag{15.50}$$

or

$$\mathbf{U} = \mathbf{VT} \tag{15.51}$$

Here is an example: Given the three control points that define a second- degree Bézier curve, we can find three new control points that define the segment of the curve from u_i to u_j in terms of a new parameter v, which spans the unit interval, $0 \le v \le 1$. The control points that define the original curve are (in matrix form)

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} \tag{15.52}$$

and the control points that define the truncated segment are

$$\mathbf{P}' = \begin{bmatrix} \mathbf{p}'_0 \\ \mathbf{p}'_1 \\ \mathbf{p}'_2 \end{bmatrix} \tag{15.53}$$

So for a second-degree Bézier curve, we have (similar to the matrix Equation 15.37 for a cubic curve)

$$\mathbf{p}(u) = \mathbf{UMP} \tag{15.54}$$

where $\mathbf{U} = [u^2 \quad u \quad 1]$ and

$$\mathbf{M} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{15.55}$$

Because $\mathbf{U} = \mathbf{VT}$, from Equation 15.50, we can write

$$\mathbf{p}(v) = \mathbf{VTMP} = \mathbf{VMP'} \tag{15.56}$$

This means that

$$\mathbf{MP'} = \mathbf{TMP} \tag{15.57}$$

or

$$\mathbf{P'} = \mathbf{M}^{-1}\mathbf{TMP} \tag{15.58}$$

where $\mathbf{V} = [v^2 \quad v \quad 1]$, and from Equation 15.49 for $n = 2$

$$\mathbf{T} = \begin{bmatrix} \Delta u_i^2 & 0 & 0 \\ 2u_i \Delta u_i & \Delta u_i & 0 \\ u_i^2 & u_i & 1 \end{bmatrix} \tag{15.59}$$

where the inverse of $\mathbf{M}$ is

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{15.60}$$

Substituting appropriately into Equation 15.58 yields the new control points

$$\mathbf{p}'_0 = (1 - u_i)^2\mathbf{p}_0 + 2u_i(1 - u_i)\mathbf{p}_1 + u_i^2\mathbf{p}_2 \tag{15.61}$$

$$\mathbf{p}'_1 = (1 - u_i)(1 - u_j)\mathbf{p}_0 + (-2u_iu_j + u_j + u_i)\mathbf{p}_1 + u_iu_j\mathbf{p}_2 \tag{15.62}$$

$$\mathbf{p}'_2 = (1 - u_j)^2\mathbf{p}_0 + 2u_j(1 - u_j)\mathbf{p}_1 + u_j^2\mathbf{p}_2 \tag{15.63}$$

15.6 Composite Bézier Curves

We can join two or more Bézier curves together, end-to-end, to create longer, more complex curves. These curves are called *composite curves*. We usually want a smooth transition from one curve to the next. One way to do this is to make sure that the tangent lines of the two curves meeting at a point are collinear. In Figure 15.9 a second-degree curve defined by control points $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ smoothly blends with a cubic curve defined by control points $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, because the control points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_0$, and $\mathbf{q}_1$ are collinear, and $\mathbf{p}_2 = \mathbf{q}_0$.

If we differentiate Equation 15.15 with respect to u and rearrange terms, we find

$$\frac{d\mathbf{p}(u)}{du} = (-3 + 6u - 3u^2)\mathbf{p}_0 + (3 - 12u + 9u^2)\mathbf{p}_1 + (6u - 9u^2)\mathbf{p}_2 + 3u^2\mathbf{p}_3 \tag{15.64}$$

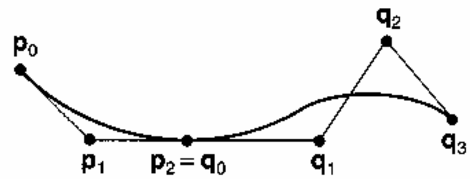


Figure 15.9 *Joining two Bézier curves.*

Evaluating this equation at $u = 0$ and $u = 1$ produces

$$\frac{d\mathbf{p}(0)}{du} = 3(\mathbf{p}_1 - \mathbf{p}_0) \tag{15.65}$$

and

$$\frac{d\mathbf{p}(1)}{du} = 3(\mathbf{p}_3 - \mathbf{p}_2) \tag{15.66}$$

This tells us that the tangent line at $u = 0$ is indeed determined by the vector $\mathbf{p}_1 - \mathbf{p}_0$, and at $u = 1$ by $\mathbf{p}_3 - \mathbf{p}_2$. These are, of course, the first and last edges of the control polygon for the cubic Bézier curve. We can obtain similar results for other Bézier curves.

Exercises

15.1 Construct enough points on the Bézier curve, whose control points are $\mathbf{p}_0 = (4, 2)$, $\mathbf{p}_1 = (8, 8)$, and $\mathbf{p}_2 = (16, 4)$, to draw an accurate sketch.
   a. What degree is the curve?
   b. What are the coordinates at $u = 0.5$?

15.2 The partition-of-unity property of Bernstein basis functions states that $\sum_i B_{i,n}(u) = 1$. Show that this is true for $n = 2$.

15.3 At what value of u is $B_{i,3}(u)$ a maximum?

15.4 Find $\mathbf{M}^{-1}\mathbf{TM}$ (Equation 15.58) for a second-degree curve.

15.5 Find $\mathbf{T}$ for $n = 3$ (Equation 15.49).