CHAPTER 10

PLANES

The mathematics and geometry of planes pervades computer graphics, geometric modeling, computer-aided design and manufacturing, scientific visualization, and many more applications. This chapter looks at various ways to define them, including the normal form and the three-point form, the relationship between a point and plane, and plane intersections.

10.1 Algebraic Definition

Euclidean geometry defines a plane in space as the locus of points equidistant from two fixed points. The resulting plane is the perpendicular bisector of the line joining the two points. We call this definition a *demonstrative* or *constructive* definition. Computer-graphics and geometric-modeling applications require a more quantitative definition, because these planes may be bounded by polygons to form the polyhedral facets representing the surface of some modeled object or the plane of projection of some view of an object, and so on.

The classic algebraic definition of a plane is the first step toward quantification. The implicit Cartesian equation of a plane is

$$Ax + By + Cz + D = 0 \tag{10.1}$$

This is a linear equation in x, y, and z. By assigning numerical values for each of the constant coefficients A, B, C, and D, we define a specific plane in space. It is an arbitrary plane because, depending on the values of the coefficients, we can give it any orientation. Figure 10.1 shows only that part of a plane that is in the positive x, y, and z octant. The three bounding lines are the intersections of the arbitrary plane with the principal planes.

If the coordinates of any point satisfy Equation 10.1, then the point lies on the plane. We can arbitrarily specify any two coordinates x and y and determine the third coordinate z by solving this equation.

We obtain a more restricted version of this general equation by setting one of the coefficients equal to zero. If we set C equal to zero, for example, we obtain

$$Ax + By + D = 0 \tag{10.2}$$

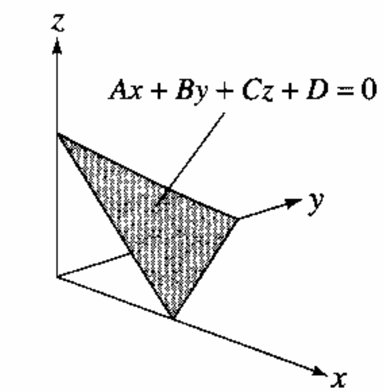

Figure 10.1 *Part of a plane.*

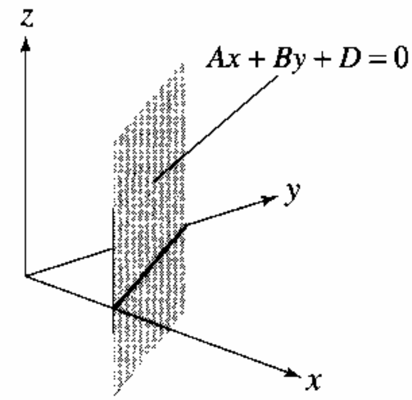

Figure 10.2 *A plane perpendicular to the x, y plane.*

This looks a lot like the equation of a line in the x, y plane. However, in three dimensional space this really means that the z coordinate of any arbitrary point can take on any value. The remaining coordinates are, of course, constrained by the relationship imposed by the equation. All planes defined this way are perpendicular to the principal plane identified by the two restricted coordinate values. In the case of Equation 10.2, the plane is always perpendicular to the x, y plane. Furthermore, the equation defines the line of intersection between the plane and the principal x, y plane (Figure 10.2).

If two of the coefficients equal zero, say $A = B = 0$, we obtain

$$Cz + D = 0 \quad \text{or} \quad z = \frac{D}{C} = k \tag{10.3}$$

where k is a constant determined by D/C. This is a plane perpendicular to the z axis and intersecting it at $z = k$ (Figure 10.3).

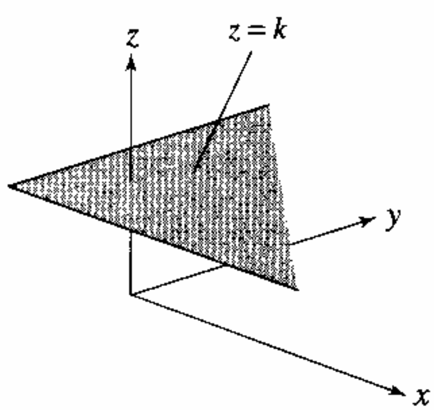

Figure 10.3 *A plane perpendicular to the z axis.*

10.2 Normal Form

A plane can also be defined by giving its *normal* (perpendicular) distance N from the origin and the direction cosines of the line defined by N, namely, d_x, d_y, d_z. These are also the direction cosines, or direction numbers, of the plane. This is all we need to define the intersections a, b, and c of the plane with each coordinate axis. To determine a, b, and c, we observe that $O\mathbf{p}_N\,\mathbf{p}_a$ is a right triangle; because N is perpendicular to the plane, any line in the plane through $\mathbf{p}_N$ is necessarily perpendicular to N (Figure 10.4). Therefore,

$$a = \frac{N}{d_x}, \quad b = \frac{N}{d_y}, \quad c = \frac{N}{d_z} \tag{10.4}$$

This yields the three points $\mathbf{p}_a = (a, 0, 0)$, $\mathbf{p}_b = (0, b, 0)$, and $\mathbf{p}_c = (0, 0, c)$. Now we can write the implicit form of the plane equation in terms of the distance N and its direction cosines as

$$d_x x + d_y y + d_z z - N = 0 \tag{10.5}$$

We can easily verify Equation 10.5 by setting any two of the coordinates equal to zero, say $y = z = 0$, to obtain the third coordinate value, in this case $x = N/d_x = a$. In general, the expressions $d_x = A$, $d_y = B$, $d_z = C$, and $N = D$ are true only if $A^2 + B^2 + C^2 = 1$. However, given $Ax + By + Cz + D = 0$, we find that

$$N = \frac{D}{\sqrt{A^2 + B^2 + C^2}} \tag{10.6}$$

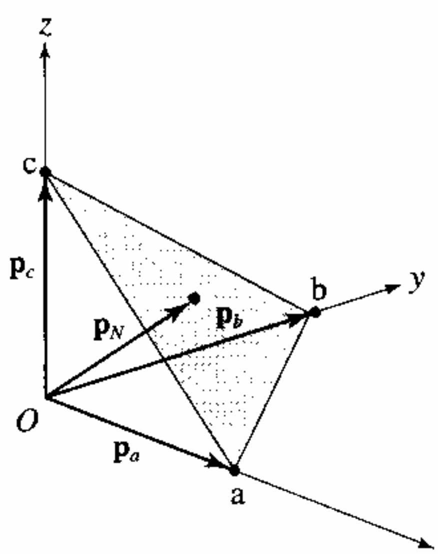

Figure 10.4 *Normal form.*

and

$$d_x = \frac{A}{\sqrt{A^2+B^2+C^2}}$$
$$d_y = \frac{B}{\sqrt{A^2+B^2+C^2}} \tag{10.7}$$
$$d_z = \frac{C}{\sqrt{A^2+B^2+C^2}}$$

Finally, if two planes have the same direction cosines, then they are parallel.

10.3 Plane Defined by Three Points

Another way to define a plane is by specifying three noncollinear points in space. We demonstrate this as follows: We assume that the coordinates of these three points are $\mathbf{p}_1 = (x_1, y_1, z_1)$, $\mathbf{p}_2 = (x_2, y_2, z_2)$, and $\mathbf{p}_3 = (x_3, y_3, z_3)$. Then we let $A' = A/D$, $B' = B/D$, and $C' = C/D$.

The implicit equation of a plane becomes

$$A'x + B'y + C'z + 1 = 0 \tag{10.8}$$

Using this equation and the coordinates of the three points, we can write

$$\begin{aligned} A'x_1 + B'y_1 + C'z_1 + 1 &= 0 \\ A'x_2 + B'y_2 + C'z_2 + 1 &= 0 \\ A'x_3 + B'y_3 + C'z_3 + 1 &= 0 \end{aligned} \tag{10.9}$$

Now we have three equations in three unknowns: A', B', and C', which we can readily solve. Two points are not sufficient to determine A', B', and C', and a fourth point would not necessarily fall on the plane.

If the three points are the points of intersection of the plane with the coordinate axes $\mathbf{p}_a = (a, 0, 0)$, $\mathbf{p}_b = (0, b, 0)$, and $\mathbf{p}_c \mp (0, 0, c)$, then the plane equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{10.10}$$

10.4 Vector Equation of a Plane

There are at least four ways to define a plane using vectors. We will discuss one of them. Others are thoroughly discussed in Section 1.8. The normal form of the vector equation is easy to express using vectors. All we must do is express the perpendicular distance from the origin to the plane in question as a vector, $\mathbf{n}$. From this the other properties follow. For example, since the components of a unit vector are also its direction cosines, we have

$$\hat{n}_x = d_x, \quad \hat{n}_y = d_y, \quad \hat{n}_z = d_z \tag{10.11}$$

where $\hat{\mathbf{n}}$ is the unit vector corresponding to $\mathbf{n}$. This means we can rewrite Equations 10.4 as follows:

$$a = \frac{|\mathbf{n}|}{\hat{n}_x}, \quad b = \frac{|\mathbf{n}|}{\hat{n}_y}, \quad c = \frac{|\mathbf{n}|}{\hat{n}_z} \tag{10.12}$$

where $|\mathbf{n}|$ is the length of $\mathbf{n}$.

We can also find the vector normal form using the implicit form of Equation 10.1 and Equations 10.6 and 10.7. Thus,

$$|\mathbf{n}| = \frac{D}{\sqrt{A^2 + B^2 + C^2}} \tag{10.13}$$

and

$$\begin{aligned} \hat{n}_x &= \frac{A}{\sqrt{A^2 + B^2 + C^2}} \\ \hat{n}_y &= \frac{B}{\sqrt{A^2 + B^2 + C^2}} \\ \hat{n}_z &= \frac{C}{\sqrt{A^2 + B^2 + C^2}} \end{aligned} \tag{10.14}$$

10.5 Point and Plane Relationships

Many problems in computer graphics and geometric modeling require us to determine on which side of a plane we find a given point $\mathbf{p}_T$. For example, we may want to know if the point is inside or outside a solid model whose bounding faces are planes. First, we define some convenient reference point $\mathbf{p}_R$ not on the plane (Figure 10.5). Then we use the implicit form of the plane equation, $f(x, y, z) = Ax + By + Cz + D$, and compute $f(x_R, y_R, z_R)$ and $f(x_T, y_T, z_T)$. Where the point lies depends on the

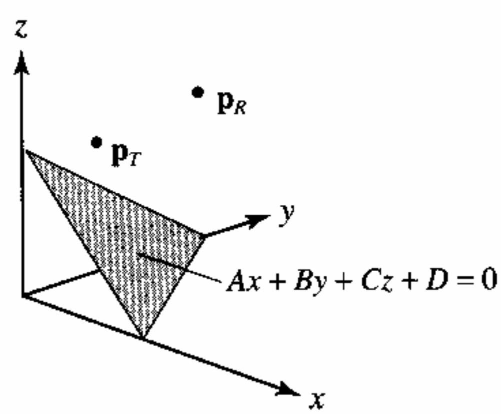

Figure 10.5 *Point and plane relationships.*

following conditions:

1. If $f(x_T, y_T, z_T) = 0$, then $\mathbf{p}_T$ is on the plane.
2. If $f(x_T, y_T, z_T) > 0$ and $f(x_R, y_R, z_R) > 0$, then $\mathbf{p}_T$ is on the same side of the plane as $\mathbf{p}_R$.
3. If $f(x_T, y_T, z_T) < 0$ and $f(x_R, y_R, z_R) < 0$, then $\mathbf{p}_T$ is on the same side of the plane as $\mathbf{p}_R$.
4. If none of the conditions above are true, then $\mathbf{p}_T$ is on the opposite side of the plane relative to $\mathbf{p}_R$.

10.6 Plane Intersections

If a line segment and plane intersect, they intersect at a point $\mathbf{p}_I$ common to both (Figure 10.6). We assume that the line segment is defined by parametric equations whose endpoint coordinates are (x_0, y_0, z_0) and (x_1, y_1, z_1). To find the coordinates of the point of intersection $\mathbf{p}_I$ we must solve the following four equations in four unknowns:

$$\begin{aligned} &Ax_I + By_I + Cz_I + D = 0 \\ &x_I = (x_1 - x_0)u_I + x_0 \\ &y_I = (y_1 - y_0)u_I + y_0 \\ &z_I = (z_1 - z_0)u_I + z_0 \end{aligned} \tag{10.15}$$

We then solve these equations for u_I:

$$u_I = \frac{-(Ax_0 + By_0 + Cz_0 + D)}{A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)} \tag{10.16}$$

If $u_I \in [0, 1]$, then we solve for x_I, y_I, and z_I. If $Ax_0 + By_0 + Cz_0 + D = 0$ and $Ax_1 + By_1 + Cz_1 + D = 0$, then the line lies in the plane. If $u_I = \infty$, that is, if $A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0) = 0$, then the line is parallel to the plane and does not intersect it.

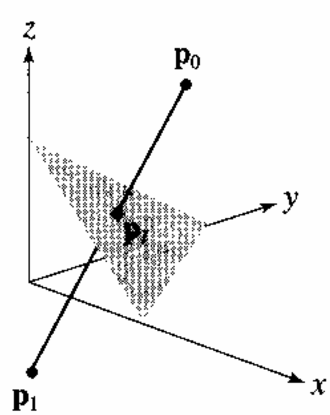

Figure 10.6 *Line and plane intersection.*

A somewhat more difficult problem is that of finding the intersection between two planes P_1 and P_2. We begin with the set of implicit equations of the two planes, solving them as two simultaneous equations in three unknowns:

$$A_1x + B_1y + C_1z + D_1 = 0$$
$$A_2x + B_2y + C_2z + D_2 = 0 \tag{10.17}$$

Knowing the basic characteristics of possible solutions to a problem helps us design more reliable and efficient algorithms. This system of equations has three possible solutions, and we can state them in terms of the geometry of intersecting planes:

1. The planes do not intersect. (They are parallel.)
2. They intersect in one line.
3. The planes are coincident.

Two unbounded nonparallel planes intersect in a straight line (Figure 10.7), and the extra unknown is more like an extra degree of freedom. Moreover, it indicates that the solution is a line and not a point (which requires a third equation).

Assigning a value to one of the variables, say $z = z_I$, reduces the system to two equations in two unknowns:

$$A_1x + B_1y = -(C_1z_I + D_1)$$
$$A_2x + B_2y = -(C_2z_I + D_2) \tag{10.18}$$

Solving these for x and y, we obtain one point on the line of intersection: $\mathbf{p}_I = (x_I, y_I, z_I)$.

Next, we assign a new value to z, say $z = z_J$, to obtain

$$A_1x + B_1y = -(C_1z_J + D_1)$$
$$A_2x + B_2y = -(C_2z_J + D_2) \tag{10.19}$$

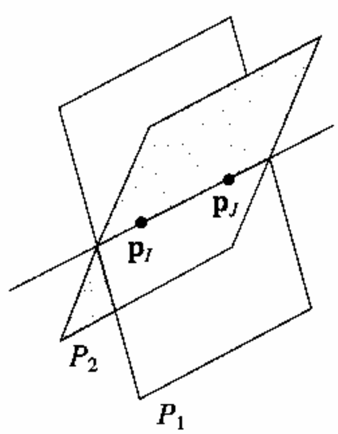

Figure 10.7 *Intersection of two planes.*

Again we solve for x and y to produce another point on the line of intersection: $\mathbf{p}_J = (x_J, y_J, z_J)$. These two points $\mathbf{p}_I$ and $\mathbf{p}_J$ are all we need to define the line of intersection in parametric form:

$$\begin{aligned} x &= (x_J - x_I)u + x_I \\ y &= (y_J - y_I)u + y_I \\ z &= (z_J - z_I)u + z_I \end{aligned} \tag{10.20}$$

Exercises

10.1 Given the three points $\mathbf{p}_1 = (a, 0, 0)$, $\mathbf{p}_2 = (0, b, 0)$, $\mathbf{p}_3 = (0, 0, c)$, show that the equation of the plane containing these points can be written as $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

10.2 Find the geometric relationships between the following points and the plane $y = -3$, where the origin is the reference point.

a. $\mathbf{p}_1 = (4, 2, 5)$
b. $\mathbf{p}_2 = (3, -3, 0)$
c. $\mathbf{p}_3 = (-2, 4, 6)$
d. $\mathbf{p}_4 = (5, -4, 3)$
e. $\mathbf{p}_5 = (-3, -7, 8)$

10.3 Find the geometric relationships between the following points and the plane $4y - 3z - 24 = 0$, where the origin is the reference point.

a. $\mathbf{p}_1 = (6, 2, 4)$
b. $\mathbf{p}_2 = (-3, 6, 0)$
c. $\mathbf{p}_3 = (8, 0, -8)$
d. $\mathbf{p}_4 = (2, 2, -9)$
e. $\mathbf{p}_5 = (-1, 7, -1)$

10.4 Find the point of intersection between each of the five lines, whose endpoints are given, and the plane $3x + 4y + z = 24$. First find the parametric equations for each line for $u \in [0, 1]$, and then find the value of the parametric variable corresponding to the intersection point, if it exists.

a. $\mathbf{p}_0 = (4, 0, 4)$, $\mathbf{p}_1 = (4, 6, 4)$
b. $\mathbf{p}_0 = (10, 0, 2)$, $\mathbf{p}_1 = (10, 2, 2)$
c. $\mathbf{p}_0 = (10, -10, 2)$, $\mathbf{p}_1 = (10, 2, 2)$
d. $\mathbf{p}_0 = (0, 0, 24)$, $\mathbf{p}_1 = (8, 0, 0)$
e. $\mathbf{p}_0 = (0, 9, 0)$, $\mathbf{p}_1 = (0, 0, 25)$